LINEAR CIRCUIT ANALYSIS (EED) – U.E.T. TAXILA 07

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INTRODUCTION

Applying Kirchhoff's laws to purely resistive circuits results in algebraic equations.

While applying laws to RC and RL circuits produces differential equations, which are more difficult to solve than algebraic equations.

The differential equations resulting from analyzing RC and RL circuits are of the first order.

Hence, the circuits are collectively known as first-order circuits.

INTRODUCTION

A First-Order Circuit is characterized by a first-order differential equation.

There are two ways to excite the RC and RL circuits.

The first way is by initial conditions of the storage elements in the circuits, so called source-free circuits.

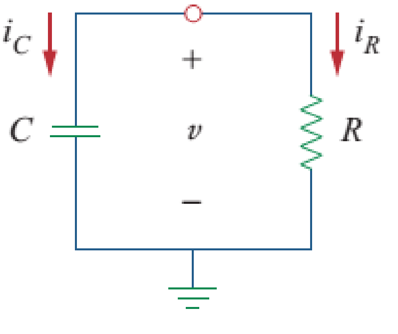
The second way of exciting first order circuits is by independent sources and these sources will be DC sources.

SOURCE-FREE RC CIRCUIT

A source-free RC circuit occurs when its DC source is suddenly disconnected.

The energy already stored in the capacitor is released to resistors.

Consider circuit;



SOURCE-FREE RC CIRCUIT

Since capacitor is initially charged and voltage $v(t)$ across capacitor at time $t=0$ is;

$$v(0) = V_0$$

The corresponding value of energy stored;

$$w(0) = \frac{1}{2}CV_0^2$$

Applying KCL at top node of circuit;

$$i_C + i_R = 0$$

SOURCE-FREE RC CIRCUIT

By definition; $i_C = C\,dv/dt$ $\quad i_R = v/R$

Putting values;

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

The first-order differential equation;

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

SOURCE-FREE RC CIRCUIT

Integrating both sides;

$$\ln v = -\frac{t}{RC} + \ln A$$

$$\ln \frac{v}{A} = -\frac{t}{RC}$$

Taking power of e;

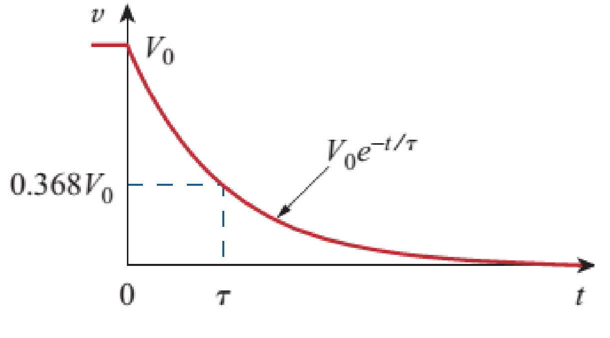
$$v(t) = Ae^{-t/RC}$$

$$v(t) = V_0 e^{-t/RC}$$

SOURCE-FREE RC CIRCUIT

This shows that the voltage response of RC circuit is an exponential decay of the initial voltage.

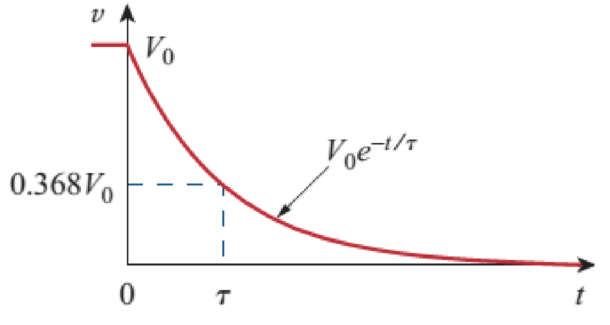
The Natural Response of a circuit refers to the behavior (in terms of voltage or current) of the circuit itself, with no external sources of excitation.



SOURCE-FREE RC CIRCUIT

The Time Constant of a circuit is the time required for the response to decay to a factor of *1/e* or *36.8* percent of its initial value.

$$V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368 V_0 \qquad \tau = RC$$



SOURCE-FREE RC CIRCUIT

Voltage across capacitor;

$$v(t) = V_0 e^{-t/\tau}$$

Current though resistor;

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau}$$

Power dissipated in resistor;

$$p(t) = v i_R = \frac{V_0^2}{R} e^{-2t/\tau}$$

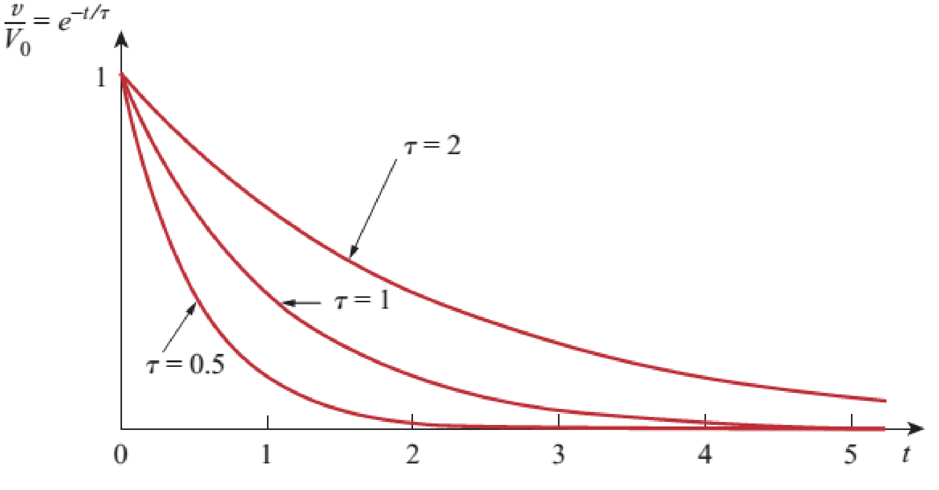
SOURCE-FREE RC CIRCUIT

Energy absorbed by the resistor;

$$w_R(t) = \int_0^t p\,dt = \int_0^t \frac{V_0^2}{R} e^{-2t/\tau}\,dt$$

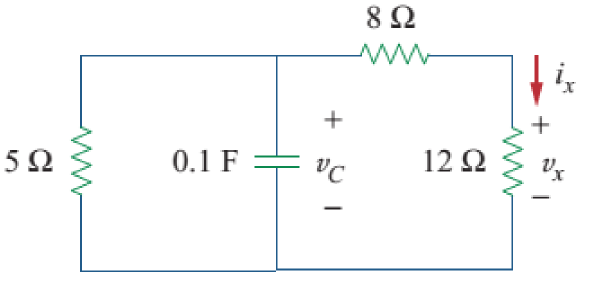
$$= -\frac{\tau V_0^2}{2R} e^{-2t/\tau}\Big|_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}), \qquad \tau = RC$$

SOURCE-FREE RC CIRCUIT



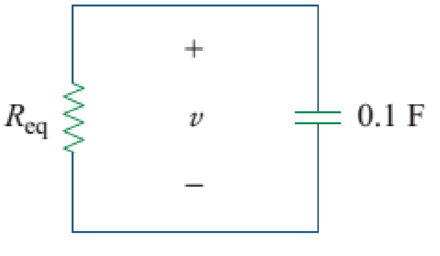
PROBLEMS

If $v_c(0)=15V$, find v_c, v_x and i_x for $t>0$.



PROBLEMS

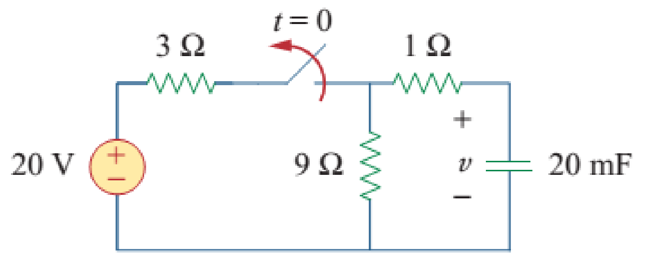
Analogous circuit;



($v_c=15e^{-2.5t}$, $v_x=9e^{-2.5t}$, $i_x=0.75e^{-2.5t}$)

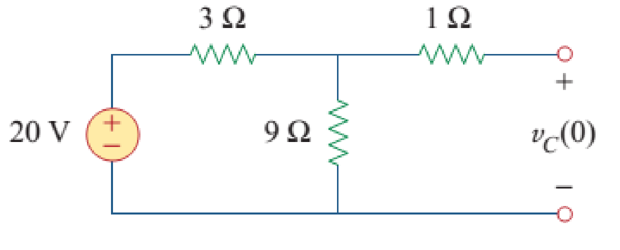
PROBLEMS

The switch in the circuit has been closed for a long time, and it is opened at $t=0$. Find v(t) for $t>0$ and initial stored energy?



PROBLEMS

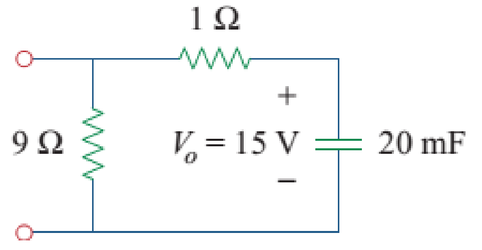
For $t<0$ the switched is closed;



(V_o=15V)

PROBLEMS

For $t>0$ the switched is opened;



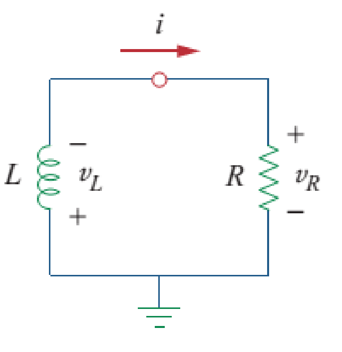
$(v(t)=15e^{-5t}, w_c(0)=2.25J)$

SOURCE-FREE RL CIRCUIT

A source-free RL circuit occurs when its DC source is suddenly disconnected.

The energy already stored in the inductor is released to resistors.

Consider circuit;



SOURCE-FREE RL CIRCUIT

Since inductor is initially charged and current $i(t)$ through inductor at time $t=0$ is;

$$i(0) = I_0$$

The corresponding value of energy stored;

$$w(0) = \frac{1}{2} L I_0^2$$

Applying KVL around the loop;

$$v_L + v_R = 0$$

SOURCE-FREE RL CIRCUIT

By definition; $v_L = L\, di/dt$ $\qquad$ $v_R = iR$

Putting values;

$$L\frac{di}{dt} + Ri = 0$$

$$\frac{di}{dt} + \frac{R}{L} i = 0$$

Rearranging and integrating;

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L}\, dt$$

SOURCE-FREE RL CIRCUIT

Integrating both sides;

$$\ln i \Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t$$

$$\ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0 \qquad \ln \frac{i(t)}{I_0} = -\frac{Rt}{L}$$

Taking power of e;

$$i(t) = I_0 e^{-Rt/L}$$

SOURCE-FREE RL CIRCUIT

This shows that the current response of RL circuit is an exponential decay of the initial current.

The Natural Response of a circuit refers to the behavior (in terms of voltage or current) of the circuit itself, with no external sources of excitation.

Time constant;

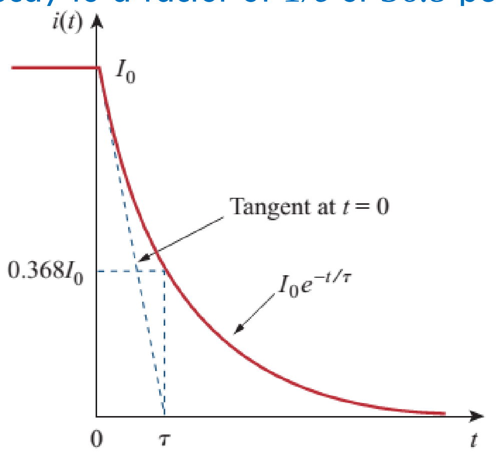
$$\tau = \frac{L}{R}$$

Current through inductor;

$$i(t) = I_0 e^{-t/\tau}$$

SOURCE-FREE RL CIRCUIT

The Time Constant of a circuit is the time required for the response to decay to a factor of $1/e$ or 36.8 percent of its initial value.



SOURCE-FREE RL CIRCUIT

Current through inductor;

$$i(t) = I_0 e^{-t/\tau}$$

Voltage across resistor:

$$v_R(t) = iR = I_0 R e^{-t/\tau}$$

Power dissipated in resistor;

$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

SOURCE-FREE RL CIRCUIT

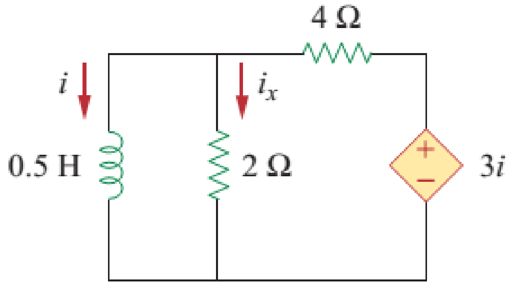
Energy absorbed by the resistor;

$$w_R(t) = \int_0^t p\,dt = \int_0^t I_0^2 R e^{-2t/\tau}\,dt = -\frac{1}{2}\tau\, I_0^2 R e^{-2t/\tau}\Big|_0^t$$

$$w_R(t) = \frac{1}{2} L\, I_0^2 (1 - e^{-2t/\tau})$$

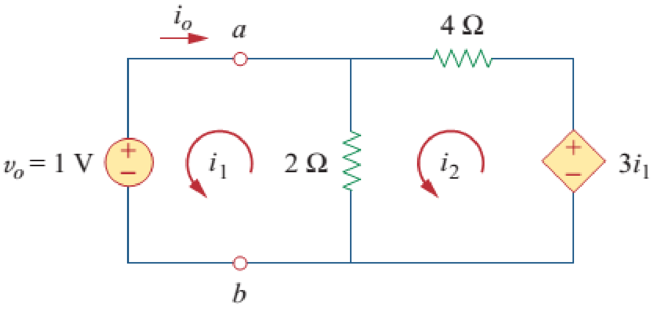
PROBLEMS

If $i(0)=10A$, find $i(t)$ and $i_x(t)$ for $t>0$.



PROBLEMS

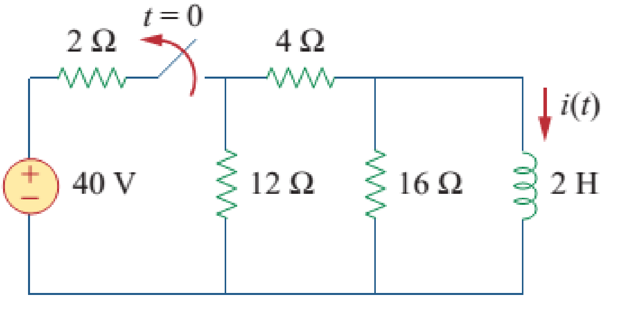
Finding equivalent resistance;



$(R_{eq}=1/3\ ohm,\ i(t)=10e^{-(2/3)t},\ i_x(t)=1.667e^{-(2/3)t})$

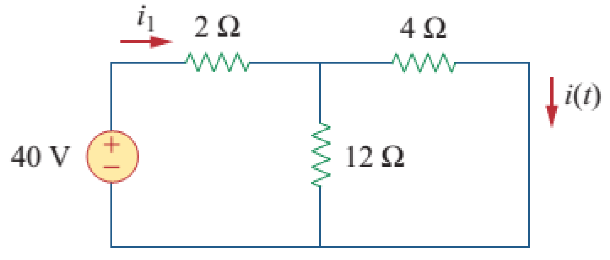
PROBLEMS

The switch in the circuit has been closed for a long time, and it is opened at $t=0$. Find i(t) for $t>0$.



PROBLEMS

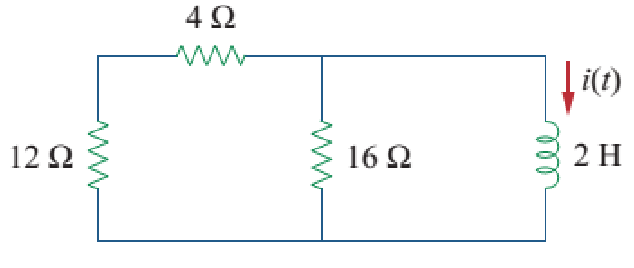
For $t<0$ the switched is closed;



(I_o=6A)

PROBLEMS

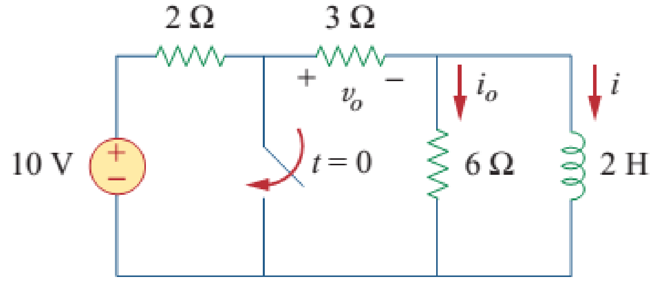
For $t>0$ the switched is opened;



($i(t)=6e^{-4t}$)

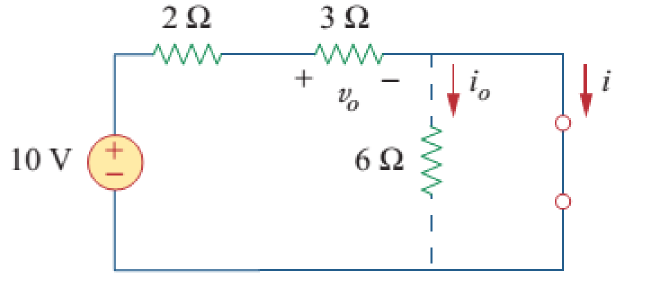
PROBLEMS

The switch in the circuit has been opened for a long time, and it is closed at $t=0$. Find i_o, v_o and i for $t>0$.



PROBLEMS

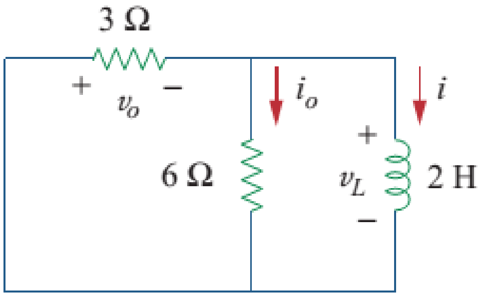
For $t<0$ the switched is opened;



(i_o=2A, v_o=6V)

PROBLEMS

For $t>0$ the switched is closed;



$(i(t)=2e^{-t},\ i_o(t)=-(2/3)e^{-t})$

PROBLEMS

For all time;

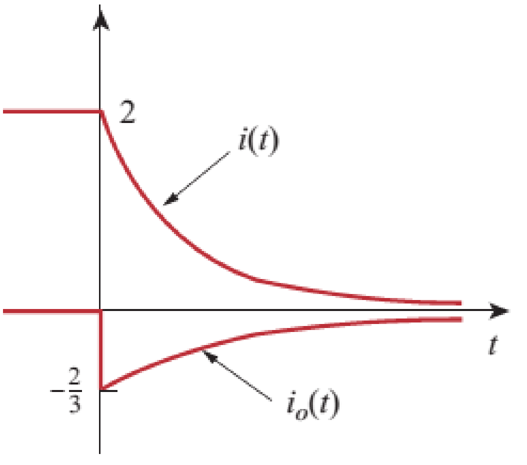
$$i_o(t) = \begin{cases} 0\ \text{A}, & t < 0 \\ -\frac{2}{3}e^{-t}\ \text{A}, & t > 0 \end{cases}$$

$$v_o(t) = \begin{cases} 6\ \text{V}, & t < 0 \\ 4e^{-t}\ \text{V}, & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2\ \text{A}, & t < 0 \\ 2e^{-t}\ \text{A}, & t \geq 0 \end{cases}$$

PROBLEMS

Plotting graphs;



SINGULARITY FUNCTIONS

Singularity functions are very useful in circuit analysis.

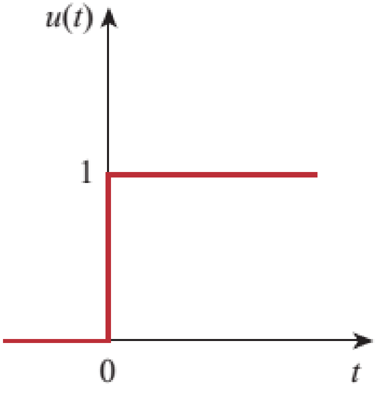
A basic understanding of singularity functions will help in understanding the response of first-order circuits to a sudden application of independent sources (DC voltage or current).

Singularity Functions are functions that either are discontinuous or have discontinuous derivatives.

The three most widely used singularity functions in circuit analysis are, unit step, unit impulse and unit ramp functions.

UNIT STEP FUNCTION

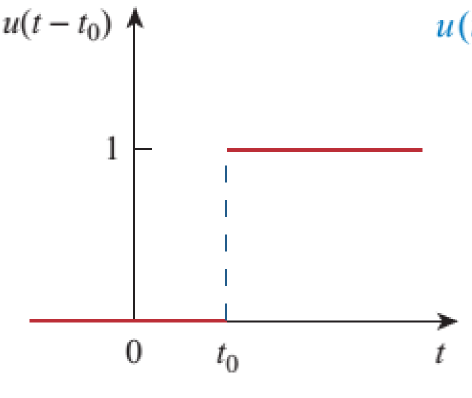
The Unit Step Function $u(t)$ is 0 for negative values of t and 1 for positive values of t.



$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

UNIT STEP FUNCTION

The unit step function may be delayed;

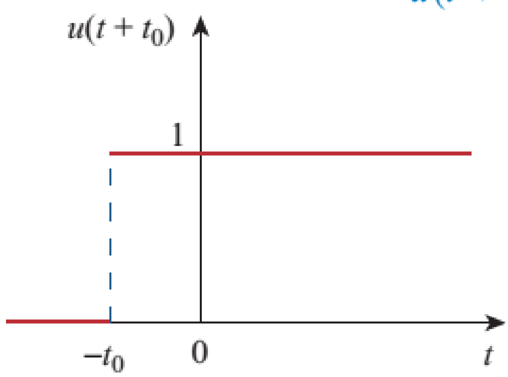


$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$

UNIT STEP FUNCTION

The unit step function may be advanced;

$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$

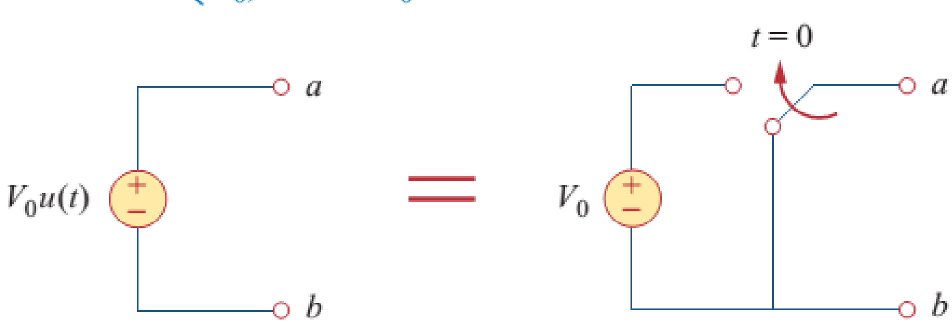


UNIT STEP FUNCTION

Voltage source may be expressed in terms of unit step function;

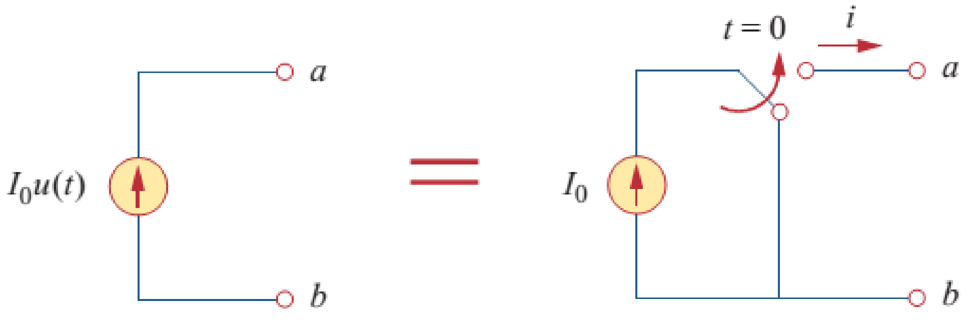
$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases}$$

$$v(t) = V_0 u(t - t_0)$$



UNIT STEP FUNCTION

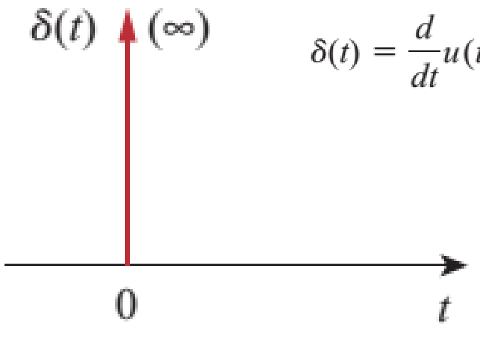
Current source may be expressed in terms of unit step function;



UNIT IMPULSE FUNCTION

The Unit Impulse Function $\delta(t)$ is zero everywhere except at $t=0$, where it is undefined.

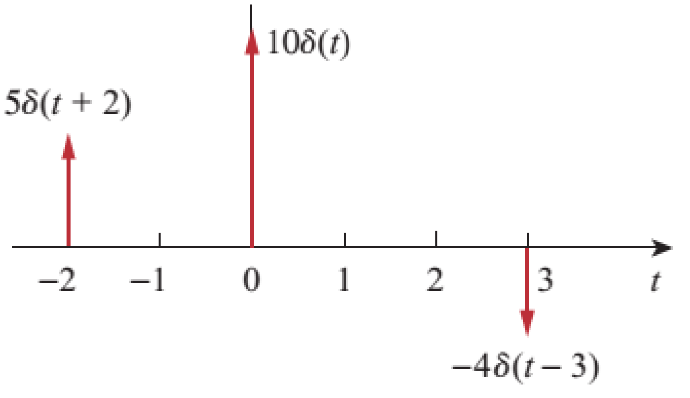
The derivative of $u(t)$ is $\delta(t)$.



$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

UNIT IMPULSE FUNCTION

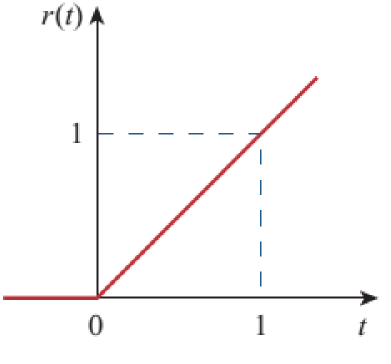
Advancing and delaying of impulse function;



UNIT RAMP FUNCTION

The Unit Ramp Function $r(t)$ is zero for negative values of t and has a unit slope for positive values of t.

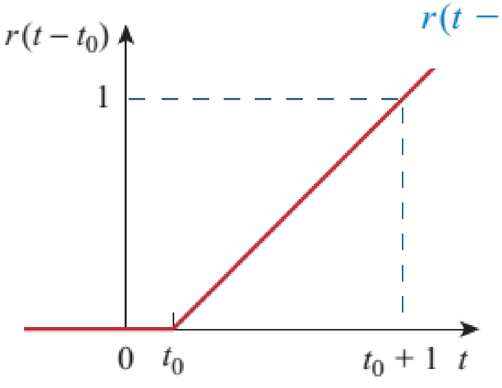
The integral of $u(t)$ is $r(t)$.



$$r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}$$

UNIT RAMP FUNCTION

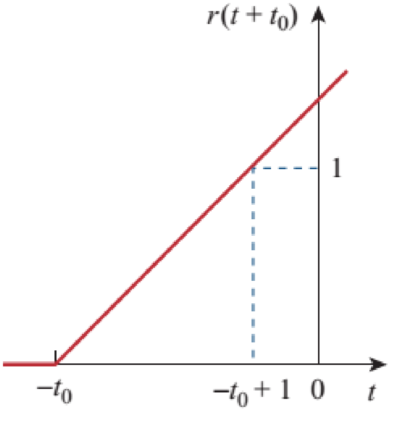
Delayed unit ramp function;



$$r(t - t_0) = \begin{cases} 0, & t \leq t_0 \\ t - t_0, & t \geq t_0 \end{cases}$$

UNIT RAMP FUNCTION

Advanced unit ramp function;



$$r(t + t_0) = \begin{cases} 0, & t \leq -t_0 \\ t + t_0, & t \geq -t_0 \end{cases}$$

SINGULARITY FUNCTIONS

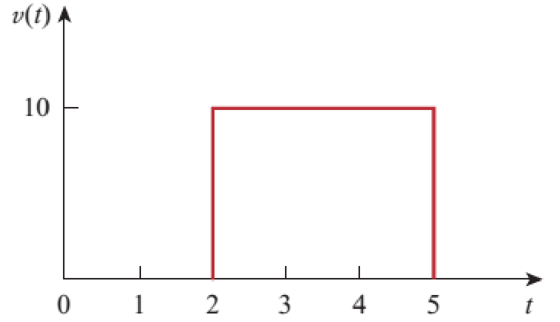
Relationships between different singularity functions;

$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt}$$

$$u(t) = \int_{-\infty}^{t} \delta(t)\,dt, \qquad r(t) = \int_{-\infty}^{t} u(t)\,dt$$

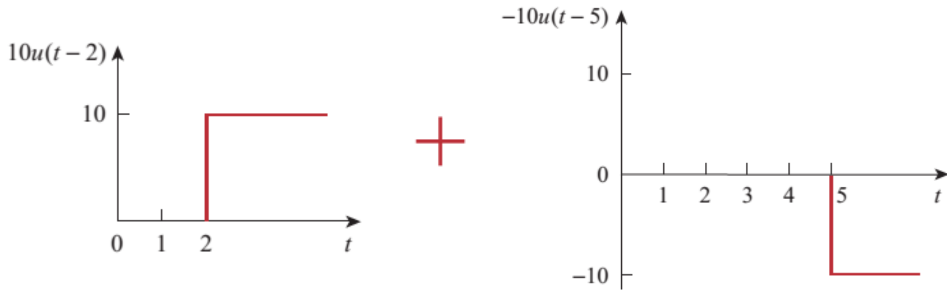
PROBLEMS

Express the voltage in terms of unit step function and also calculate its derivative.



PROBLEMS

First and second parts;

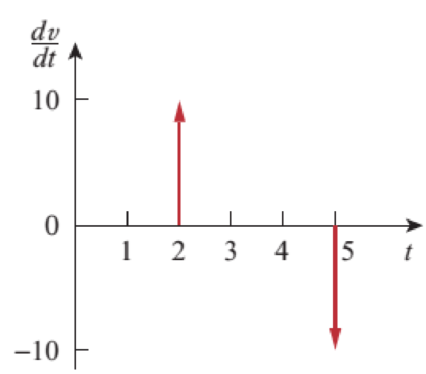


$$v(t) = 10u(t-2) - 10u(t-5) = 10[u(t-2) - u(t-5)]$$

PROBLEMS

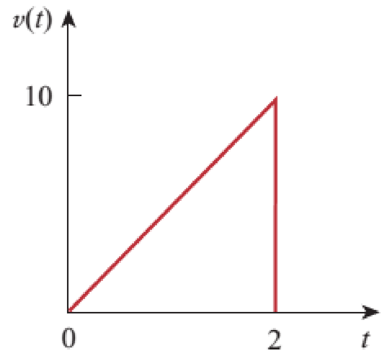
Derivative of unit step function;

$$\frac{dv}{dt} = 10[\delta(t-2) - \delta(t-5)]$$



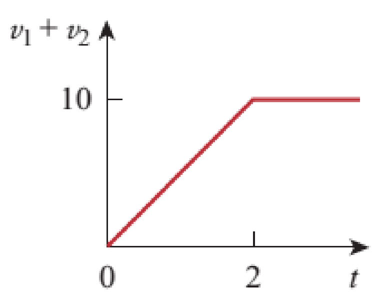
PROBLEMS

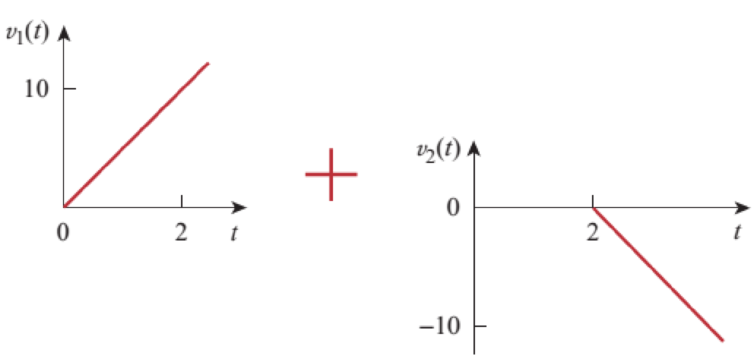
Express saw tooth function in terms of singularity functions?



PROBLEMS

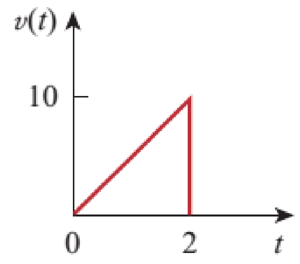
First;

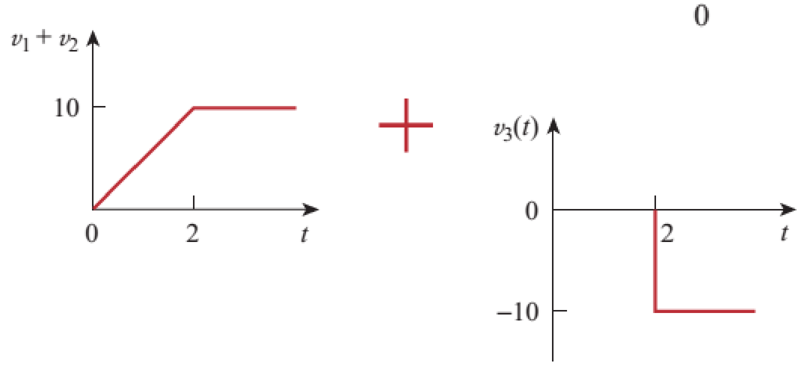




PROBLEMS

Second;





REFERENCES

Fundamentals of Electric Circuits (4th Edition)

Charles K. Alexander, Matthew N. O. Sadiku

Chapter 07 – First-Order Circuits (7.1 – 7.4)

Exercise Problems: 7.1 – 7.38

Do exercise problem yourself.

STEP RESPONSE OF AN RC CIRCUIT

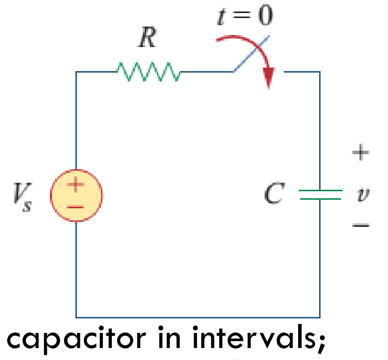
When the DC source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as step response.

The Step Response of a circuit is its behavior when the excitation is the step function, which may be a voltage or current source.

The step response is the response of the circuit due to a sudden application of a DC voltage or current source.

STEP RESPONSE OF AN RC CIRCUIT

Consider circuit;

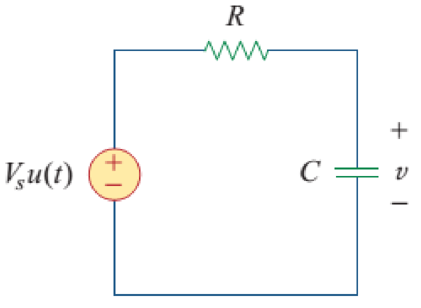


Voltage across capacitor in intervals;

$$v(0^-) = v(0^+) = V_0$$

STEP RESPONSE OF AN RC CIRCUIT

After switch is closed;



Applying KCL;

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

STEP RESPONSE OF AN RC CIRCUIT

Rearranging;

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t)$$

For $t>0$;

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

$$\frac{dv}{v - V_s} = -\frac{dt}{RC}$$

STEP RESPONSE OF AN RC CIRCUIT

Integrating;

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC}$$

Taking exponential;

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$$

STEP RESPONSE OF AN RC CIRCUIT

Rearranging;

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0$$

Voltage across capacitor;

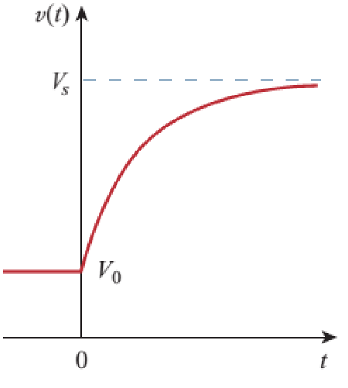
$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

This is complete response of the RC circuit to a sudden application of DC source when capacitor is charged.

STEP RESPONSE OF AN RC CIRCUIT

Response of the RC circuit when capacitor is charged;

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

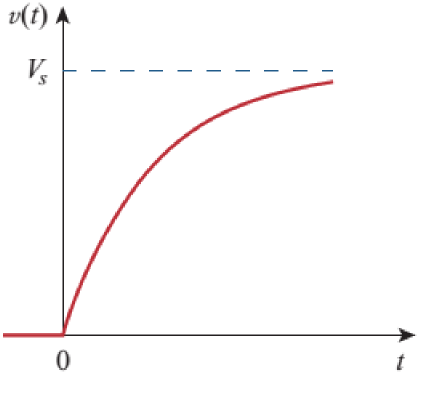


STEP RESPONSE OF AN RC CIRCUIT

Response of the RC circuit when capacitor is uncharged;

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

$$v(t) = V_s(1 - e^{-t/\tau})u(t)$$

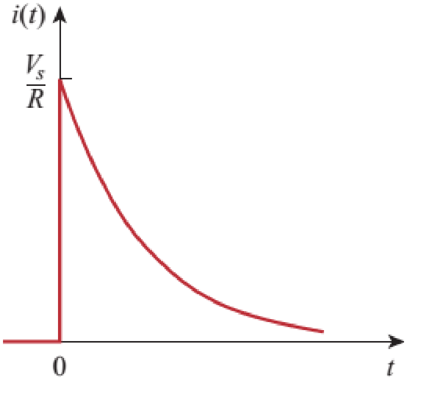


STEP RESPONSE OF AN RC CIRCUIT

Current through capacitor;

$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-t/\tau} \qquad \tau = RC$$

$$i(t) = \frac{V_s}{R}e^{-t/\tau}u(t)$$



STEP RESPONSE OF AN RC CIRCUIT

Complete response of the RC circuit;

$$\text{Complete response} = \underbrace{\text{transient response}}_{\text{temporary part}} + \underbrace{\text{steady-state response}}_{\text{permanent part}}$$

$$v = v_t + v_{ss}$$

Where;

$$v_t = (V_o - V_s)e^{-t/\tau}$$

$$v_{ss} = V_s$$

STEP RESPONSE OF AN RC CIRCUIT

The Transient Response is the circuit's temporary response that will die out with time.

The Steady-State Response is the behavior of the circuit a long time after an external excitation is applied.

It may also be expressed as;

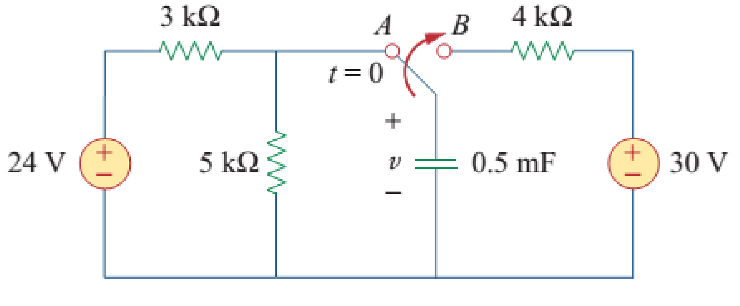
$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

Where;

1. The initial capacitor voltage $v(0)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant τ.

PROBLEMS

The switch has been in position A for a long time. At $t=0$ the switch moves to B. Find $v(t)$ for $t>0$ and calculate its value at $t=1$ s and $t=4$ s.



($v(t)=(30-15e^{-0.5t})$, 20.9V, 27.97V)

STEP RESPONSE OF AN RL CIRCUIT

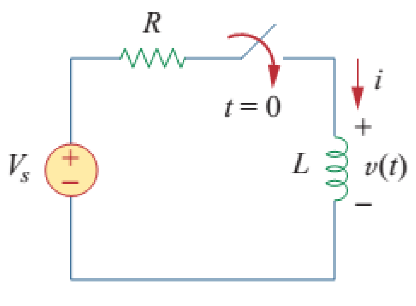
When the DC source of an RL circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as step response.

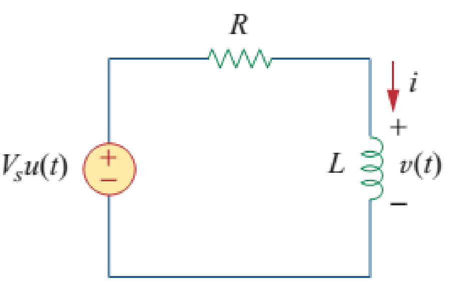
The Step Response of a circuit is its behavior when the excitation is the step function, which may be a voltage or current source.

The step response is the response of the circuit due to a sudden application of a DC voltage or current source.

STEP RESPONSE OF AN RL CIRCUIT

Consider circuit;





Let the response of RL circuit;

$$i = i_t + i_{ss}$$

STEP RESPONSE OF AN RL CIRCUIT

Transient response is always a decaying exponential;

$$i_t = Ae^{-t/\tau}, \qquad \tau = \frac{L}{R}$$

The steady state response;

$$i_{ss} = \frac{V_s}{R}$$

Substituting values;

$$i = Ae^{-t/\tau} + \frac{V_s}{R}$$

Inductor's initial current; $i(0^+) = i(0^-) = I_0$

STEP RESPONSE OF AN RL CIRCUIT

At t=0;

$$I_0 = A + \frac{V_s}{R}$$

$$A = I_0 - \frac{V_s}{R}$$

Putting value of A, inductor current;

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-t/\tau}$$

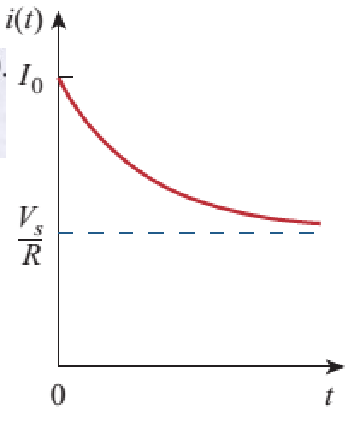
STEP RESPONSE OF AN RL CIRCUIT

Complete response;

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

Where;

1. The initial inductor current $i(0)$ at $t = 0$.
2. The final inductor current $i(\infty)$.
3. The time constant τ.



STEP RESPONSE OF AN RL CIRCUIT

If inductor is uncharged; $I_0 = 0$

$$i(t) = \begin{cases} 0, & t < 0 \\ \frac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t)$$

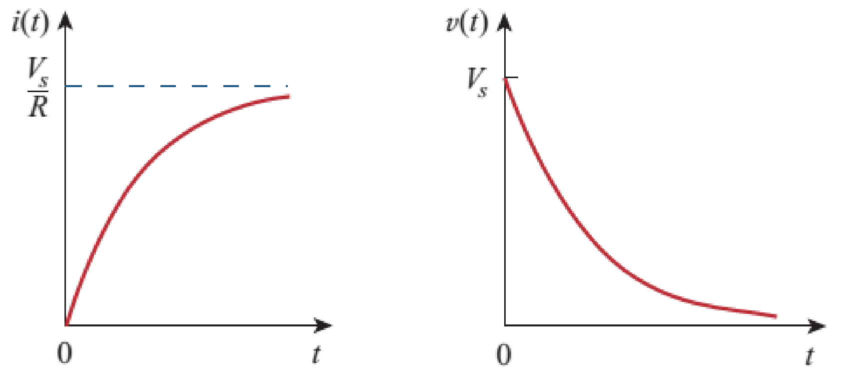
Voltage across inductor;

$$v(t) = L\frac{di}{dt} = V_s\frac{L}{\tau R}e^{-t/\tau}, \qquad \tau = \frac{L}{R}, \qquad t > 0$$

STEP RESPONSE OF AN RL CIRCUIT

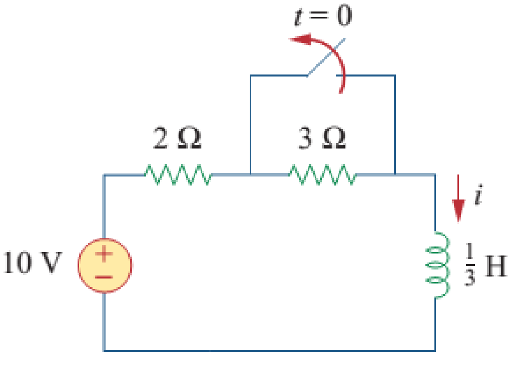
Voltage across inductor;

$$v(t) = V_s e^{-t/\tau} u(t)$$



PROBLEMS

Find $i(t)$ in the circuit for $t>0$. Assume that switch has been closed for long time.



$(i(t)=(2+3e^{-15t}))$

REFERENCES

Fundamentals of Electric Circuits (4^{th} Edition)

Charles K. Alexander, Matthew N. O. Sadiku

Chapter 07 – First-Order Circuits (7.5 – 7.6)

Exercise Problems: 7.39 – 7.65

Do exercise problem yourself.